Optimal Risk Manipulation in Asymmetric Tournament

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Abstract

We fill a gap in the literature of asymmetric tournament by exploring risk manipulating behavior of the principal. In our model with heterogeneous agents and limited liability, a principal can manipulate the level of risk so that the performance of each agent becomes less or more uncertain. Our result shows that there exists an optimal risk level from the principal's standpoint of maximizing expected profit. Our model and its implications help to explain why organizations conducting similar businesses have different performance measure standards with different precision or accuracy.

Keywords: Asymmetric Tournament; Risk Manipulation; Incentive Contract

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1 Introduction

Since the pioneering work of Lazear and Rosen (1981), many aspects of tournaments have been studied.[1] One of them, especially important in asymmetric tournaments, is the manipulation of risk such as changing the precision/accuracy of performance measure. Risk manipulation behaviors may be undertaken by either agents or principal. For instance, employees can choose between more or less risky production technologies, or work more carefully or carelessly; employers can create more comfortable or stressful workplace environments, or make performance standards to be more clear or ambiguous, or monitor the work site more strongly or weakly.

A number of studies have analyzed how agents can be benefited by influencing the mean and distribution of outcome in a tournament. See, for example, Hvide (2002), Kräkel and Sliwka (2004), and Chen (2003). However, less can be said about how the principal can gain by manipulating risk in an asymmetric tournament. O'Keefe et al. (1984) show that managers can maintain the incentive compatibility constraint by appropriately adjusting the level of monitoring precision in an asymmetric tournament. But they did not solve for an optimal level of precision. Nti (2004), also discussed in Wang (2010), shows the existence of an optimal accuracy/risk level in a contest setting when two heterogeneous agents are sufficiently different in their abilities.

The rest of the paper is organized as follows. Section 2 introduces our model setup with results presented in Section 3. In Section 4, we show simulations and numerical examples. The last section concludes.

2 Model Setup

Consider a typical setting of tournament where a principal employs two asymmetric agents, labeled as 1 and 2, and implements relative performance evaluation.[2] The output of agent *i*, $i = 1, 2$, denoted as y_i, is given by

[1] See, for example, Green and Stokey (1983), O'Keefe et al. (1984), Rosen (1986), Bhattacharya and Guasch (1988), Skaperdas and Gan (1995), Yun (1997), Moldovanu and Sela (2001), Reed (2001), and Kräkel and Sliwka (2004).

[2] We assume that it is effective for the principal to use tournament. Milgrom and Roberts (1992, pp.367-368) point out that tournaments can be the only way to provide incentives to agents in some situations, although relative

$$\begin{cases} y_1 = k + a_1 + \varepsilon_1 \\ y_2 = a_2 + \varepsilon_2 \end{cases} \tag{1}$$

where a_i is the effort level unobserved by the principal and the other agent, and ε_i is noise naturally occurred in production process and is independent and identically distributed. Without loss of generosity, we assume $\varepsilon_i \sim N\left(0, \frac{1}{2}\right)$ throughout this paper. *k* is the difference in ability of the two agents with $k > 0$ indicates the fact that agent 1 is of higher ability and hence has higher expected output than agent 2 (net of noise). We follow the literature in assuming that ability and effort are perfect substitutes.

To incorporate risk manipulation by the principal, we assume the outcome of relative performance evaluation is determined by

$$\begin{cases} \hat{y}_1 = k + a_1 + \lambda\varepsilon_1 \\ \hat{y}_2 = a_2 + \lambda\varepsilon_2 \end{cases} \tag{2}$$

where $\hat{y}_i$ is the performance of agent *i* evaluated by the principal, $\lambda(>0)$ is the coefficient of risk manipulation. The principal can manipulate λ in order to change the noise in production. As a result, $\lambda\varepsilon_i \sim N\left(0, \frac{1}{2}\lambda^2\right)$. All players can only observes $\hat{y}_i$ but not a_i or y_i.

The principal's goal is to maximize expected profit given by

$$V = E[y_1] + E[y_2] - 2w - \Delta = a_1 + a_2 - 2w - \Delta \tag{3}$$

where w and Δ are respectively competitive wage and rewards to be paid to the agents and determined endogenous by the principal in our model.

The timeline of the game is as follows. In the first stage, the principal chooses the coefficient of risk manipulation, λ. In the second stage, the principal then designs the contract (w and Δ). In the third and last stage, the two agents independently choose a_i. The game is solved backwardly.

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performance evaluation may not be Pareto improving to contracts with pay tied to individual performance (Wolfstetter, 1999, pp.307).

3 Results

We first solve for the optimal decision of the agents. From equation (2), the winning probability of agent 1 is given by

$$\begin{aligned}
&\Pr\left[k+a_1+\lambda\varepsilon_1>a_2+\lambda\varepsilon_2\right]\\
&=\Pr\left[\lambda\varepsilon_1-\lambda\varepsilon_2>a_2-a_1-k\right]\\
&=1-G\left(a_2-a_1-k\right)
\end{aligned}\tag{4}$$

where $G(\cdot)$ is the cumulative distribution function (*cdf*) of random variables $\left(\lambda\varepsilon_1-\lambda\varepsilon_2\right)\sim N\left(0,\lambda^2\right)$. For risk-neutral agents, we assume their expected utilities, u_i, to be expressed by

$$\begin{cases}
u_1=w+\left[1-G\left(a_2-a_1-k\right)\right]\Delta-\dfrac{a_1^2}{2}\\
u_2=w+G\left(a_2-a_1-k\right)\Delta-\dfrac{a_2^2}{2}
\end{cases}\tag{5}$$

where $a_i^2/2$ is the disutility of effort for the two agents. Taking the first order derivative of equation (5), we obtain the incentive compatibility constraints (IC) as

$$\begin{cases}
g\left(a_2-a_1-k\right)\Delta=a_1\\
g\left(a_2-a_1-k\right)\Delta=a_2
\end{cases}\tag{6}$$

where $g(\cdot)$ is the probability density function (*pdf*) of $\left(\lambda\varepsilon_1-\lambda\varepsilon_2\right)$. We can obtain the optimal solutions for a_i from equation (6) as

$$a_1^*=a_2^*=a^*=g\left(-k\right)\Delta\,.\tag{7}$$

Substituting in the *pdf* of normal distribution, we obtain

$$\frac{\partial a_i^*}{\partial\lambda}=\frac{\left(k^2-\lambda^2\right)}{\lambda^4\sqrt{2\pi}}\exp\left(-\frac{k^2}{2\lambda^2}\right)\Delta\,.\tag{8}$$

We establish the following lemma according to this derivative.

Lemma 1. *Given that the amount of reward is positive ($\Delta > 0$), when the two agents are homogeneous ($k = 0$), their optimal efforts (a_i^*) decrease as risk (λ) increases. However, when agents are heterogeneous ($k > 0$), the optimal efforts increase as risk increases if risk is low ($\lambda \in [0, k]$) and decrease if risk is high ($\lambda > k$).*

Proof: The result followed immediate from equation (8). ■

Lemma 1 is intuitive. When the outcome of the tournament is determined by not only effort but a little of luck, the weaker agent will have incentive to compete if he can get better luck with the principal's choice of a larger λ . But when the outcome depends too much on luck, then both agents become opportunists and hence none of them is willing to work hard.

We now investigate the second stage of the game. Substituting equation (7) into equation (3) and taking into account both agents individual rationality constraint (IR) as well as the limited liability constraint (LL), we rewrite the principal's expected profit maximization problem as

$$\begin{aligned} &\max_{\{w,\Delta\}} V = \left[2g(-k)-1\right]\Delta - 2w \\ &s.t. \quad \text{(IR1)} \quad w + \left[1 - G(-k)\right]\Delta - \frac{a^2}{2} \geq 0; \\ &\qquad\ \ \text{(IR2)} \quad w + G(-k)\Delta - \frac{a^2}{2} \geq 0; \\ &\qquad\ \ \text{(LL)} \quad w \geq 0. \end{aligned} \tag{9}$$

It is straightforward to verify that (LL) and (IR2) are binding while (IR1) is non-binding. As a result, we obtain $w = 0$ and rewrite the problem (after substituting in the solution of a^*) as

$$\begin{aligned} &\max_{\{w,\Delta\}} V = \left[2g(-k)-1\right]\Delta \\ &s.t. \quad G(-k)\Delta - \frac{\left[g(-k)\Delta\right]^2}{2} \geq 0. \end{aligned} \tag{10}$$

Solving for Δ, we obtain

$$\begin{cases} \Delta^* = 0 & \text{if} \quad 2g(-k) \le 1 \\ \Delta^* = \dfrac{2G(-k)}{g(-k)^2} & \text{if} \quad 2g(-k) > 1 \end{cases}.$$

Since $a^* = 0$ and $V^* = 0$ when $\Delta^* = 0$, we focus only on the case where $2g(-k) > 1$. For this, we have the following lemma

Lemma 2. *There exists a feasible set for the principal to extract efforts from agents in the tournament. This feasible set requires* $k < \bar{k} \equiv \lambda\sqrt{\log\left[2\left(\pi\lambda^2\right)^{-1}\right]}$ *with* $\lambda^{\max} = \sqrt{2/\pi}$ *and* $\bar{k}^{\max} = \sqrt{2/(\pi e)}$.

Proof: If $2g(-k) \le 1$, the principal prefer to choose $\Delta = 0$ (hence $a = 0$ and $V = 0$) rather than any $\Delta > 0$. The principal would like to choose $\Delta > 0$ (and extract a positive effort $a_i > 0$) if and only if $2g(-k) > 1$. Plugging the normal distribution *pdf* into the condition and simplify, we can prove the first part of Lemma 2. For $\lambda^{\max}$, notice that the value of $\bar{k}$ is constrained by $\log\left[2\left(\pi\lambda^2\right)^{-1}\right] > 0$ which in turn constraints λ and gives $\lambda \le \lambda^{\max} \equiv \sqrt{2/\pi}$. For $\bar{k}^{\max}$, we take the first order derivative of $\bar{k}$ w.r.t. λ and obtain $\bar{k}_\lambda = \dfrac{\log\left[2(\pi\lambda)^{-2}\right] - 1}{\sqrt{\log\left[2(\pi\lambda)^{-2}\right]}}$.[3] Setting the first order derivative to 0, we obtain $\lambda = \sqrt{2/(\pi e)}$ and hence $\bar{k}^{\max} = \sqrt{2/(\pi e)}$. ■

When $2g(-k) > 1$ is satisfied, we have

[3] Note that the second order derivative, $\bar{k}_{\lambda\lambda} = -\dfrac{2}{3}\left(\dfrac{1+\log\left[2/\left(\pi\lambda^2\right)\right]}{\lambda\log\left[2/\pi\lambda^2\right]}\right)$, is negative. This guarantees the strict concavity of $\bar{k}$ as a function of λ.

$$\Delta^* = \frac{2G(-k)}{\left[g(-k)\right]^2};$$
$$a^* = \frac{2G(-k)}{g(-k)};$$
$$V^* = \frac{4G(-k)}{g(-k)} - \frac{2G(-k)}{\left[g(-k)\right]^2}.$$

We can now proceed to the first stage of the game and solve for the optimal value of λ. Taking the first order derivative of equation (10) w.r.t. λ, we obtain

$$\frac{\partial V}{\partial \lambda} = \frac{\partial}{\partial \lambda}\left(\frac{4G(-k)}{g(-k)}\right) - \frac{\partial}{\partial \lambda}\left(\frac{2G(-k)}{\left[g(-k)\right]^2}\right) = MR - MC$$
$$= \underbrace{\frac{1}{\lambda^2}\left[4k\lambda - 4\left(k^2-\lambda^2\right)\sqrt{2\pi}\exp\left(\frac{k^2}{2\lambda^2}\right)G(-k)\right]}_{MR} \qquad (11)$$
$$-\underbrace{\frac{1}{\lambda}\left[2k\lambda\sqrt{2\pi}\exp\left(\frac{k^2}{2\lambda^2}\right) - 8\left(k^2-\lambda^2\right)\pi\exp\left(\frac{k^2}{\lambda^2}\right)G(-k)\right]}_{MC}.$$

From equation (11), we can establish the following proposition

Proposition 1. *When* $\lambda \in \left[k, \lambda^{\max}\right]$, *and parameters are in the principal's feasible set, there exists an unique* λ^* *that maximizes the expected profit of the principal.*

Proof: First, it can be verified that when $\lambda \in \left[k, \lambda^{\max}\right]$, *MR* and *MC* are both monotonic and increase in λ. When $\lambda = k$, we have $MR_{\lambda=k} = 4$ and $MC_{\lambda=k} = 2k\sqrt{2\pi e} < 2\bar{k}^{\max}\sqrt{2\pi e} = 4$. As a result, when $\lambda = k$, we have $MR > MC$. On the other end, when $\lambda = \lambda^{\max}$, $k = 0$. Substituting $\lambda^{\max}$ and $k = 0$ into equation (11), we have $MR_{\lambda=\lambda^{\max}} - MC_{\lambda=\lambda^{\max}} = -4G(0)\sqrt{2\pi} = -2\sqrt{2\pi} < 0$. As a result, $MR_{\lambda=\lambda^{\max}} < MC_{\lambda=\lambda^{\max}}$. This proves that there exists a single crossing of the two functions. ■

4 Simulation and Numerical Examples

This section provides some simulations and numerical examples of the Model. In Figure 1, the shaded area between $\bar{k}$ (the inverted-U shape curve) and the horizontal axis is the principal's feasible set, in which manipulating the risk is feasible for the principal. Principal will get zero profit at any point on curve $\bar{k}$, and get positive/negative profit in the area below/above curve $\bar{k}$. The path $\lambda^*(k)$ (in red) depicts the optimal choices of the principal given different values of k, as outlined in Proposition 1. As can be seen in Figure 1, an optimal λ with positive profit occurs only when $\lambda > k$ (under the 45 degree line). The intersection of path λ^* and the 45 degree line is $\bar{k}^{\max}\left(=\sqrt{2/(\pi e)}\right)$.

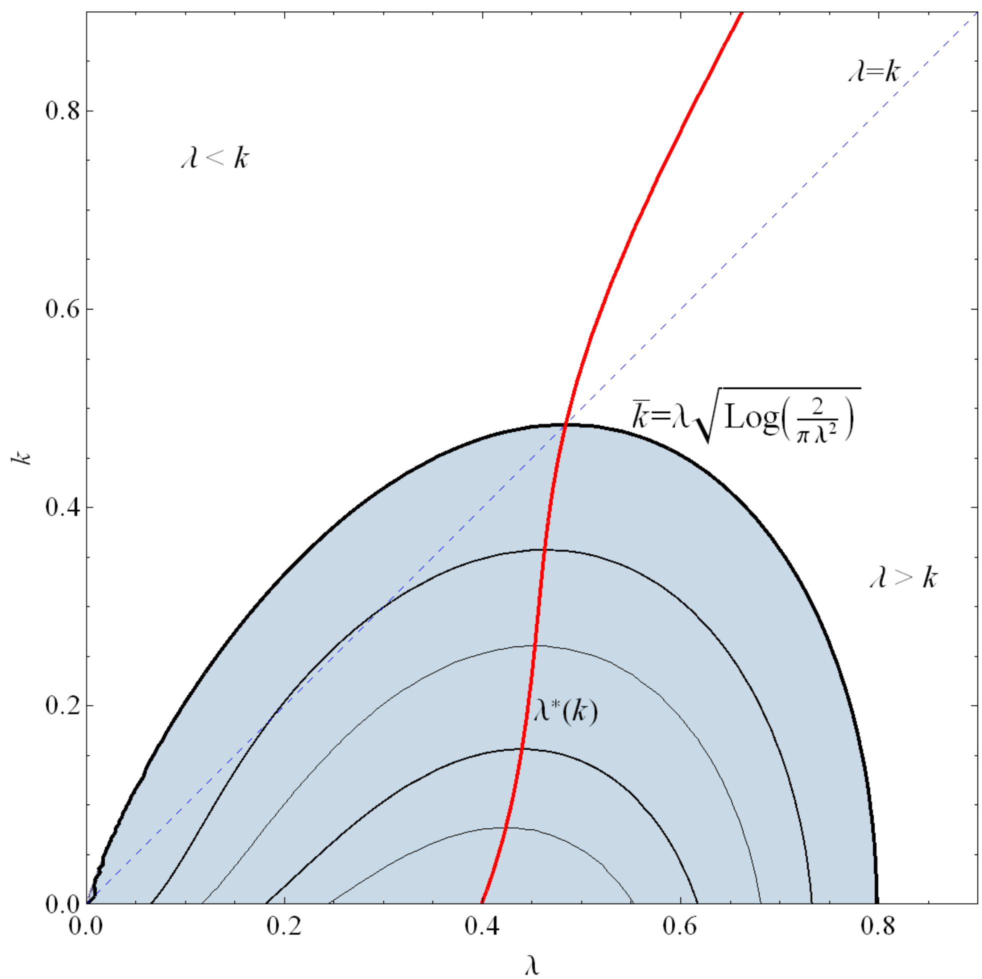

Figure 1: Feasible set of risk manipulation and optimal risk pattern

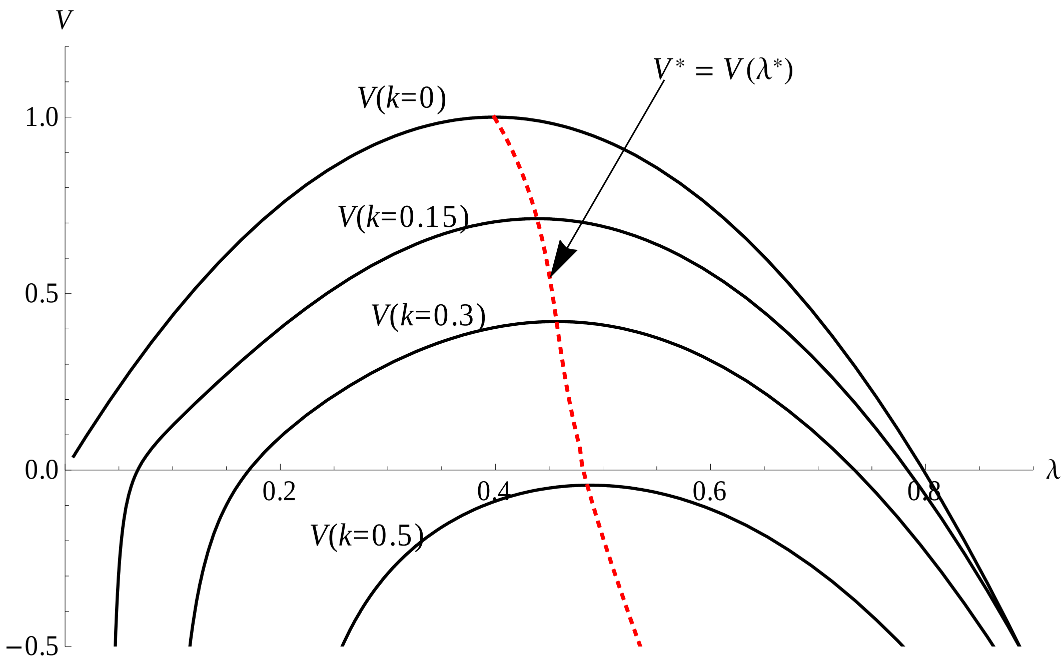

Figure 2: Expected profit of the principal

Figure 2 depicts the expected profit of the principal when $k=0$, 0.15, 0.3, and 0.5. A higher curve is corresponding to a lower value of k, namely less heterogeneity between agents. So, the larger the heterogeneity between agents, the lower the principal's expected profit. The expected profit even turns into negative when $k > \bar{k}^{\max}\left(=\sqrt{2/(\pi e)} \approx 0.484\right)$. This is shown by the curve $V(k=0.5)$.

From numerical calculation, we find that when $k=0$, $\lambda^*=1/\sqrt{2\pi} \approx 0.399$, $a_i^*=1$, and $V^*=1$. This is exactly the solution to a tournament with risk neutral agents without limited liability (see Wolfstetter, 1999, Chapter 12), which is the most efficient outcome in tournaments. Put differently, under limited liability, when the principal can optimally manipulate risk, we can achieve the same efficient result as without limited liability.

Furthermore, when $k=0.15$, $\lambda^*=0.438$, $a_i^*=0.854$, and $V^*=0.712$; when $k=0.3$, $\lambda^*=0.457$, $a_i^*=0.726$, and $V^*=0.420$. Therefore, as the agents become more heterogeneous, the principal chooses a higher risk level, which distorted the production even further (with a lower expected profit). The dashed line shows the path of V^*, which is corresponding to the path of λ^* in Figure 1, at different levels of *k*.

5 Conclusion

Built upon a typical tournament setting, our model allows a principal to optimally manipulate risk in relative performance evaluation. Our model shows that, when the two heterogeneous agents are not significantly different, there always exists a unique optimal level of risk manipulation. As the difference of ability between the two agents gets larger, the optimal manipulated risk level becomes higher. At the mean time, efforts chosen by agents become lower so that the principal receives a lower expected payoff. These results help to explain why managers tend to choose workers with similar ability and why organizations conducting similar business have very different performance measure standards with different precision or accuracy.

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